$$
\begin{aligned}
p+q+r+s+t&=t,\\
q+2r+3s+4t&=s,\\
r+3s+6t&=r,\\
s+4t&=q,\\
t&=p,
\end{aligned}
$$

whence

$$p=1,\ q=2,\ r=-1,\ s=-2,\ t=1.$$

Substituting these values in the general binary equation, we have

$$x^4+2x^3y-x^2y^2-2xy^3+y^4=24n\pm 1 \text{ or } 4(24n\pm 1).$$

The same general properties, Theorems I and II, may be predicated of this quartic as of the above cubic, inclusive of its enlargement for the composite numbers *G. C. D.* equal $6m \pm 1$. Examples: —

Assume x, y equal any two numbers prime to each other, $x=4, y=31$. Evidently

$$4^4+2.4^3.31-4^2.31^2-2.4.31^3+31^4=24.28085+1,\ n=28085.$$

Assume $x=5, y=15$. *G. C. D.* $=5$, $m=1$. Evidently

$$625+3750-5625-33750+50625=24.651+1,\ n=651.$$

SOLUTIONS OF EXERCISES.

2

THREE closely connected tanks, T_1, T_2, T_3, contain Q_1 gallons of water, Q_2 gallons of vinegar, Q_3 gallons of brandy, respectively. A flow is set up from T_1 through T_2 to T_3 and back to T_1 at the rate of 1 gallon per second. The liquids are assumed to mix instantaneously, and the lengths of the connecting pipes are neglected. Show how to calculate the amount of water in each tank at the end of t seconds. [*De Volson Wood.*]

SOLUTION.

Let x_1, x_2, and x_3 be the quantities of water in the three tanks at the time t;

then, since $x_1 \div Q_1$ is the quantity of water contained in a unit of the mixture in the tank T_1, etc., we have, putting for convenience a_1, a_2, and a_3 for the reciprocals of Q_1, Q_2, and Q_3, the differential equations

$$\frac{dx_1}{dt} = a_3x_3 - a_1x_1, \text{ etc.};$$

or, putting D for $\frac{d}{dt}$ symbolically,

$$(D + a_1)\,x_1 - a_3x_3 = 0, \tag{1}$$
$$(D + a_2)\,x_2 - a_1x_1 = 0, \tag{2}$$
$$(D + a_3)\,x_3 - a_2x_2 = 0. \tag{3}$$

Solving for x_1 we find

$$\begin{vmatrix} D + a_1 & & -a_3 \\ -a_1 & D + a_2 & \\ & -a_2 & D + a_3 \end{vmatrix} x_1 = 0$$

or

$$D[D^2 + (a_1 + a_2 + a_3)D + a_1a_2 + a_2a_3 + a_3a_1]\,x_1 = 0. \tag{4}$$

The same equation exists for x_2 and x_3, the values differing from that of x_1 only in the constants of integration. The complete integral of (4) may be written in the form

$$a_1x_1 = Ae^{\lambda_2 t} + Be^{\lambda_1 t} + C, \tag{5}$$

where λ_1 and λ_2 are the roots of the quadratic

$$\lambda^2 + (a_1 + a_2 + a_3)\,\lambda + a_1a_2 + a_2a_3 + a_3a_1 = 0; \tag{6}$$

then from (1)

$$a_3x_3 = \frac{\lambda_1 + a_1}{a_1}\,Ae^{\lambda_1 t} + \frac{\lambda_2 + a_1}{a_1}\,Be^{\lambda_2 t} + C, \tag{7}$$

whence from (3)

$$a_2x_2 = \frac{\lambda_1 + a_3}{a_3}\,\frac{\lambda_1 + a_1}{a_1}\,Ae^{\lambda_1 t} + \frac{\lambda_2 + a_3}{a_3}\,\frac{\lambda_2 + a_1}{a_1}\,Be^{\lambda_2 t} + C. \tag{8}$$

To determine the constants we have when $t = 0$, $x_1 = Q_1$, $x_2 = 0$, $x_3 = 0$; hence (5), (7), and (8) give

$$1 = A + B + C,$$
$$0 = \frac{\lambda_1 + a_3}{a_1}\,A + \frac{\lambda_2 + a_1}{a_1}\,B + C,$$
$$0 = \frac{\lambda_1 + a_3}{a_3}\,\frac{\lambda_1 + a_1}{a_1}\,A + \frac{\lambda_2 + a_3}{a_3}\,\frac{\lambda_2 + a_1}{a_1}\,B + C;$$

from which we obtain

$$-1 = \frac{\lambda_1}{a_1} A + \frac{\lambda_2}{a_1} B,$$

and

$$0 = \frac{\lambda_1}{a_3} \frac{\lambda_1 + a_1}{a_1} A + \frac{\lambda_2}{a_3} \frac{\lambda_2 + a_1}{a_1} B.$$

Whence

$$A = \frac{a_1 (\lambda_2 + a_1)}{\lambda_1 (\lambda_1 - \lambda_2)},$$
$$B = \frac{a_1 (\lambda_1 + a_1)}{\lambda_2 (\lambda_2 - \lambda_1)},$$
$$C = \frac{\lambda_1 \lambda_2 + a_1 (\lambda_2 + \lambda_1 + a_1)}{\lambda_1 \lambda_2},$$

which by (6) reduces to

$$C = \frac{a_2 a_3}{a_1 a_2 + a_2 a_3 + a_3 a_1},$$

Hence

$$x_1 = \frac{\lambda_2 + a_1}{\lambda_1 (\lambda_1 - \lambda_2)} e^{\lambda_1 t} + \frac{\lambda_1 + a_1}{\lambda_2 (\lambda_2 - \lambda_1)} e^{\lambda_2 t} + \frac{C}{a_1},$$
$$x_3 = \frac{(\lambda_1 + a_1)(\lambda_2 + a_1)}{a_3 \lambda_1 (\lambda_1 - \lambda_2)} e^{\lambda_1 t} + \frac{(\lambda_2 + a_1)(\lambda_1 + a_1)}{a_3 \lambda_2 (\lambda_2 - \lambda_1)} e^{\lambda_2 t} + \frac{C}{a_3},$$
$$x_2 = \frac{(\lambda_1 + a_3)(\lambda_1 + a_1)(\lambda_2 + a_1)}{a_2 a_3 \lambda_1 (\lambda_1 - \lambda_2)} e^{\lambda_1 t} + \frac{(\lambda_2 + a_3)(\lambda_2 + a_1)(\lambda_1 + a_1)}{a_2 a_3 \lambda_2 (\lambda_2 - \lambda_1)} e^{\lambda_2 t} + \frac{C}{a_2},$$

where λ_1 and λ_2 are

$$-\frac{1}{2}(a_1 + a_2 + a_3) \pm \frac{1}{2} \sqrt{(a_1^2 + a_2^2 + a_3^2 - 2a_1 a_2 - 2a_2 a_3 - 2a_3 a_1)},$$

and C is as above. Adding these values, it will be found that

$$x_1 + x_2 + x_3 = \frac{1}{a_1} = Q_1,$$

as evidently should be the case.

[*Wm. Woolsey Johnson.*]

3

FIND the radius of a sphere which, let fall into a given conical wine-glass full of water, will displace the maximum or minimum quantity of water.

[*Benjamin Alvord.*]

SOLUTION.

Let x be the radius of the sphere, y the distance of its centre from the surface of the water. The volume displaced will be

$$V = \frac{\pi}{3}(x-y)^2(2x+y).$$

But if a is the radius of the top, b the depth, and c the slant height of the cone

$$y = \frac{cx-ab}{a};$$

whence

$$V = \frac{\pi}{3a^3}(ax-cx+ab)^2(2ax+cx-ab).$$

This will have its maximum or minimum value when

$$f'(x) = 2(ax-cx+ab)(2ax+cx-ab)(a-c)+(ax-cx+ab)^2(2a+c) \\ = 0.$$

Two cases then arise.

1. The first is given by putting

$$ax-cx+ab=0.$$

The corresponding value of V is a minimum; but the circle of contact of the sphere with the cone is above the top of the glass; so that this solution must be rejected.

2. The second is given by putting

$$2(2ax+cx-ab)(a-c)+(ax-cx+ab)(2a+c)=0,$$

whence

$$x = \frac{abc}{(c+2a)(c-a)}.$$

The corresponding value of V is a maximum, as is shown by substitution in $f''(x)$. [*W. F. C. Hasson.*]

4

On one of the bounding radii of a given quadrant, radius r, a semicircle is drawn, radius $\frac{1}{2}r$, the semicircle being within the quadrant. Find the average area of the circle touching the other bounding radius of the quadrant and the arc of the semicircle. [*Artemas Martin.*]

SOLUTION BY THE PROPOSER.

Let x and y be the co-ordinates of a point on the locus of the centres of

the tangent circles, the axis of x being the diameter of the semicircle, and that of y being the other bounding radius of the quadrant; then we have

$$\left(\frac{1}{2}r - x\right)^2 + y^2 = \left(\frac{1}{2}r + x\right)^2;$$

whence

$$y^2 = 2rx. \tag{1}$$

Therefore the centres of all the circles are located on the arc of a parabola whose vertex is at the origin of co-ordinates. The *largest* circle is that which also touches the arc of the quadrant. We then have the additional equation

$$y^2 + x^2 = (r - x)^2,$$

whence

$$y^2 = r^2 - 2rx. \tag{2}$$

From this and (1) we get

$$y = \frac{1}{2}r\sqrt{2},$$

which is the superior limit of y.

Let s = length of the parabolic arc, estimated from the origin of co-ordinates, and $\varDelta$ = the average area required; then, the area of the circle being $\pi x^2 = \dfrac{\pi y^4}{4r^2}$,

$$\varDelta = \frac{\displaystyle\int_{y=0}^{y=\frac{1}{2}r\sqrt{2}} \frac{\pi y^4}{4r^2}\,.\,ds}{\displaystyle\int_{y=0}^{y=\frac{1}{2}r\sqrt{2}} ds}$$

But by (1)

$$ds = \sqrt{(dx^2 + dy^2)} = \frac{1}{r}\sqrt{(y^2 + r^2)}\,dy.$$

$$\therefore \varDelta = \frac{\displaystyle\frac{\pi}{4r^3}\int_0^{\frac{1}{2}r\sqrt{2}} y^4\sqrt{(r^2 + y^2)}\,dy}{\displaystyle\frac{1}{r}\int_0^{\frac{1}{2}r\sqrt{2}} \sqrt{(r^2 + y^2)}\,dy}$$

$$= \frac{1}{32}\pi r^2 \left\{ 1 - \frac{\sqrt{3}}{\sqrt{3} + 2\log\left(\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{6}\right)} \right\}.$$

5

THREE lines are drawn from a point within a square to three corners of the square. The lengths of these lines are given. Find a side of the square.

[*J. M. Greenwood.*]

SOLUTION I.

Let a, b, c, be the lengths of the lines. On the sides of the right angle A lay off AP, AQ, each equal to a, and join PQ. On PQ as base with bc as sides construct the triangle PQR, and join AR. Then AR is the side of the required square.

[*Alexander Evans.*]

SOLUTION II.

Refer the points to the diagonals of the square as axes, and let x,y be its coordinates. Let l be the length of the half-diagonal. Then

$$x^2 + (l - y)^2 = a^2,$$
$$x^2 + (l + y)^2 = b^2,$$
$$y^2 + (l \pm x)^2 = c^2;$$

whence for l the biquadratic

$$8l^4 - 4(a^2 + b^2)l^2 + a^4 + b^4 + 2c^4 - 2a^2c^2 - 2b^2c^2 = 0.$$

The side of the square is $l\sqrt{2}$.

[*E. W. Morley.*]

SOLUTION III.

Let AB, BC be sides of the square, and M the point. Put $AB = s$; MA, MB, $MC = a, b, c$. Then

$$\cos ABM = \frac{s^2 + b^2 - a^2}{2bs},$$
$$\cos MBC = \frac{s^2 + b^2 - c^2}{2bs};$$

and, as these angles are complementary,

$$(s^2 + b^2 - a^2)^2 + (s^2 + b^2 - c^2)^2 = 4b^2s^2,$$

a quadratic in s^2 from which the length of the side is readily computed.

[*L. G. Barbour.*]

8

IF an ellipse and a rectangular hyperbola have the same centre, and the hyperbola passes through the focus of the ellipse, then at the point of intersec-

tion of the curves the ellipse makes equal angles with the hyperbola and the central radius. [*H. A. Newton.*]

SOLUTION I.

The rectangular hyperbola may always be regarded as generated by two equal (projective) pencils of opposite directions of rotation. In the present case the two foci F_1, F_2 of the ellipse, being points of the hyperbola, may be assumed as the centres of such pencils. To completely determine the position of the hyperbola, it only remains to assume an arbitrary ray F_2A of the pencil F_2 as corresponding to the ray F_1F_2 of the pencil F_1. The asymptotes will then be parallel to the bisectors of the angles formed by F_2F_1 and F_2A.

If P be one of the points of intersection of the two curves, and Q the point where the bisector of $\angle AF_2F_1$ meets F_1P, we evidently have

$$\angle PF_1F_2 = \angle PF_2A,$$

and hence

$$\angle PQF_2 = \angle PF_2Q.$$

A line through P parallel to F_2Q (*i. e.* parallel to the asymptote) will therefore make equal angles with PF_1 and PF_2, that is, will be tangent to the ellipse in P. Thus we see that the tangent and normal of the ellipse in P are parallel to the asymptotes of the hyperbola. This, in connection with the property of the hyperbola that the segment of the tangent intercepted by the asymptotes is bisected in the point of contact, proves the proposition. [*Alexander Ziwet.*]

SOLUTION II.

Let the equations of the ellipse and rectangular hyperbola be

$$x^2 + \frac{y^2}{m} = c, \tag{1}$$

$$x^2 - y^2 - 2pxy = c\,(1 - m), \tag{2}$$

where (2), being satisfied by $x = \sqrt{c(1 - m)}$, $y = 0$, passes through the focus of of the ellipse. Combining (1) and (2), we get for their point of intersection

$$x^2 - m\,(c - x^2) - 2px\sqrt{m\,(c - x^2)} = c\,(1 - m). \tag{3}$$

Now, if $x'y'$ denote this point, we have for the equation to the central radius

$$y = \frac{y'}{x'}x; \tag{4}$$

for the equation to the tangent to the ellipse,

$$y = -\frac{mx'}{y'}\,x + mc; \tag{5}$$

for the equation to the tangent to the hyperbola,

$$xx' - p(x'y + y'x) - yy' = c(1 - m)$$

or

$$y = \frac{x' - py'}{y' + px'}\, x - \frac{c(1 - m)}{y' + px'}. \tag{6}$$

Let α equal the angle between the central radius and the tangent to the ellipse, and β equal the angle between the tangents to the ellipse and the hyperbola; then

$$\tan\beta = \frac{x'y' - py'^2 + mx'y' + mpx'^2}{y'^2 + px'y' - mx'^2 + mpx'y'},$$

$$\tan\alpha = \frac{mx'^2 + y'^2}{(m - 1)x'y'}.$$

That $\tan\alpha$ may equal $\tan\beta$, we must have, discarding the primes,

$$(mx^2 + y^2)(y^2 + pxy - mx^2 + mpxy) = (mxy - xy)(xy - py^2 + mxy + mpx^2),$$

which reduces to

$$y^4 + 2mpx^3y + 2mpxy^3 - m^2x^4 - m^2x^2y^2 + x^2y^2 = 0, \tag{7}$$

or, substituting for y its value from (1),

$$m^2(c - x^2)^2 + 2mpx^3\sqrt{m(c - x^2)} + 2m^2px(c - x^2)\sqrt{m(c - x^2)},$$
$$- m^2x^4 - m^3x^2(c - x^2) + mx^2(c - x^2) = 0. \tag{8}$$

From (3),

$$2px\sqrt{m(c - x^2)} = (1 + m)x^2 - c.$$

Substituting in (8), we get

$$m^2(c - x^2)^2 + mx^2(x^2 + mx^2 - c) + m^2(c - x^2)(x^2 + mx^2 - c) - m^2x^4 - m^3cx^2$$
$$+ m^3x^4 + mcx^2 - mx^4 = 0,$$

an identity. Therefore $\tan\alpha = \tan\beta$, and $\alpha = \beta$. Q. E. D.

[*S. M. Barton.*]

EXERCISES.

13

A THIN inextensible cord in which the density of the material increases in geometric progression, as the distance from one end increases in arithmetic pro-